TRANSIENT ANALYSIS AND APPLICATIONS OF MARKOV REWARD PROCESSES

THESIS

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Abstract

In this thesis, the problem of computing the cumulative distribution function (cdf) of the random time required for a system to first reach a specified reward threshold when the rate at which the reward accrues is controlled by a continuous-time stochastic process is considered. This random time is a type of first passage time for the cumulative reward process. The major contribution of this work is a simplified, analytical expression for the Laplace-Stieltjes transform of the cdf in one dimension rather than two. The result is obtained using two techniques: i) by converting an existing partial differential equation to an ordinary differential equation with a known solution, and ii) by inverting an existing two-dimensional result with respect to one of the dimensions. The results are applied to a variety of real-world operational problems using one-dimensional numerical Laplace inversion techniques and compared to solutions obtained from numerical inversion of a two-dimensional transform, as well as those from Monte-Carlo simulation. Inverting one-dimensional transforms is computationally more expedient than inverting two-dimensional transforms, particularly as the number of states in the governing Markov process increases. The numerical results demonstrate the accuracy with which the one-dimensional result approximates the first passage time probabilities in a comparatively negligible amount of the time.

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1. Introduction

1.1 Background

In this thesis, the transient analysis of Markov reward processes (MRPs) will be considered. Markov reward processes are used to model a wide variety of real-world systems. Some of these include multiprocessor computer systems, transportation systems, and manufacturing systems. By modelling such systems as MRPs, analysts are able to study performance measures which would be difficult to evaluate otherwise. These performance measures might be completion times of a job, total cost associated with a product, or the time required to transport a product from its origin to its ultimate destination. Given the competitive environment that exists in today's world, the ability to study and understand these performance measures can be critical to an organization's success. For these reasons, Markov reward processes have been studied extensively in recent years.

There are several different strategies that can be used to analyze a Markov reward process. First, one could conduct a steady-state analysis of the process in which the behavior of the system is considered as time tends to infinity. In other situations, it may be appropriate to conduct a transient analysis of the system wherein the cumulative value of the reward process is sought for some finite time value.

1.2 Problem Definition and Methodology

The primary objective of this thesis is to investigate the probability distribution of the accumulated reward in a Markov reward process. The accumulated reward is directly influenced by a random process that can be modelled as a continuous-time Markov chain (CTMC), $\{Z(t) : t \geq 0\}$. The state of $Z(t)$ controls the rate at which the reward linearly accumulates. It will be assumed that the rates are all positive, and therefore the accumulated reward up to time t is a nondecreasing stochastic process. The properties of this process will be discussed in Chapter 3. Finding the distribution of the reward also provides the distribution of $T(x)$, the random time to accumulate reward level x.

Clearly, a study of $T(x)$ will require transient analysis of the system. It makes little sense to study the system as $t \to \infty$ when the interest is specifically in the system from time 0 to $T(x)$. To obtain numerical results for such a problem, Laplace and Laplace-Stieltjes transforms will be utilized. In some cases, once the solutions are found in the transform space, the Laplace transform inversion tables can be utilized to provide an exact solution. However, it is more likely that numerical inversion of the transform will be needed to provide an approximate answer to the problem.

The existing literature reveals that these types of problems are generally solved in two-dimensional transform space. However, two-dimensional inversion is not always an easy (or efficient) task. The contribution of this thesis is to reduce the dimensionality of these problems, so that solutions can be obtained in one dimension as opposed to two. Two methods will be developed to find these one-dimensional solutions, and numerical examples will be provided demonstrating the utility of the solution on a variety of operational problems.

1.3 Thesis Outline

The remainder of this thesis is organized in the following manner. Chapter 2 reviews the previous literature on the topic. Not only does this review address many of the known fundamental results for reward processes, but it also gives examples of their usefulness through some real-world applications. Chapter 3 provides a formal mathematical description of the problem, derives both one- and two-dimensional results, and discusses numerical inversion of Laplace transforms. In Chapter 4, numerical examples will be demonstrated, providing a means to compare the one- and two-dimensional methods. Finally, Chapter 5 provides conclusions, recommendations, and future work.

2. Review of the Literature

In this chapter, a review of some relevant literature on Markov reward models is presented. The first section introduces Markov reward models, and discuss the advantages of Markov modelling over other alternatives. Section 2.2 briefly examines steady-state analysis of Markov reward processes in the literature. Finally, in Section 2.3, the transient analysis of Markov reward processes is considered. In addition, this section provides examples from the literature of specific systems modelled as Markov reward processes, and discusses performance measures and the methods with which they are obtained.

2.1 Markov Reward Processes

Markov reward models, also referred to as Markov reward processes (MRP), allow analysts to accurately model systems that evolve stochastically over time. A Markov reward model consists of two elements: a structure-state process and a reward structure. The structure-state process is assumed to be an irreducible, positive recurrent continuous-time Markov chain (CTMC) with a specified state space, S. The reward structure consists of the rates at which reward (or cost) is accrued when the structure-state process occupies each state in S. Thus, the reward rate depends explicitly on the state of the system.

One example of such a system is presented by Kulkarni [13]. Consider a machine that can be in one of two states: it is either functioning or it is not functioning. If it is functioning, it fails after a random amount of time which is exponentially distributed with rate μ. If it is not functioning, it is repaired after a random amount of time which is exponentially distributed with rate λ. Next, define a random variable $X(t)$, such that if $X(t) = 0$, the machine is not functioning at time t and if $X(t) = 1$, the machine is functioning at time t. Therefore, the state of the machine is described by the continuous-time Markov chain $\{X(t) : t \geq 0\}$, which is called

the structure-state process. Now suppose that the machine produces profit at rate r dollars per unit time when it is functioning. But when the machine is down, the repairs cost c dollars per unit time. This constitutes the reward structure in this model. When $X(t) = 0$, the rate is $-c$, and when $X(t) = 1$, the rate is r.

Kulkarni [13] gives another example which involves a machine shop that has two machines that are independent, identical, and have the same failure and repair rates as the machine in the previous example. This time, let $X(t)$ be the number of working machines at time t. It is easily shown that the structure-state process, $\{X(t) : t \geq 0\}$, is a continuous-time Markov chain. As one would expect, the rate at which the machine shop produces work at a specified time, t, is completely dependent on the number of functioning machines at time t. Therefore, there exist different reward rates when there are zero, one, or two machines operating. This is another example of a Markov reward process.

Once a system has been modelled analytically as a Markov reward process, the model may be analyzed to determine various performance measures, such as the expected time to complete a job, or the expected time-averaged cost. Although analytical modelling is not the only technique used to make predictions about system performance, it is often the preferred technique. Besides analytical modelling, there are two other general methods for making predictions about the performance of a system [18]. The first technique involves actual observation and measurement of performance metrics. Clearly, repeated measurements over an extended period of time on the actual system being studied would help analysts assess the behavior of a given performance measure. However, there are two major disadvantages to this technique. Unfortunately, it is often quite expensive to collect the data. Moreover, actual measurements cannot be collected if the system does not exist. As an example, if the study is performed to select the most efficient design of an assembly line, this technique cannot be utilized, since the assembly line does not exist.

Another alternative for making predictions is computer simulation. There are many computer software packages available to simulate a wide variety of systems ranging from the very simple to the very complex. But as Reibman, *et al.* [18] point out, simulation can also be expensive and may require considerable time to produce statistically significant results.

Because of the noted disadvantages of the other two techniques, analytical modelling is generally used whenever possible. There are several types of analytical models found in the literature, including combinatorial models, Markov chains, Markov renewal processes, and Markov reward processes. Markov models are able to capture complex attributes of a system's behavior [18], and will therefore be the focus of this literature review.

One of the major advantages of Markov reward processes over other types of analytical models is that they allow the analyst to combine traditional performance with reliability. In the past, models have predicted system performance under the assumption that systems operate failure-free [16]. Obviously, real-world systems cannot operate indefinitely without experiencing failures. These failures (and the associated repairs) are often modelled using reliability analysis. However, in recent years, analysts have become increasingly aware of the fact that the separation of performance and reliability models is no longer adequate [16]. This has led to an increased interest in applying Markov reward modelling to systems in which the structure-states represent various levels of degradation. This application is particularly prevalent in the area of computer systems [19].

It is evident from the literature that Markov reward processes can be an extremely useful tool. However, in building a Markov model, the analyst should consider five major aspects to maximize its usefulness [18]:

- Make the model easy to use. If the model is too complicated, the advantages of Markov modelling may be outweighed by the disadvantage of its difficulty to use.
- Minimize computation costs. A model is much more useful if does not require tremendous amounts of computation time, or computer memory.
- Error analysis. An analyst should understand the sources of error, and seek to minimize that error. Some examples of error sources are modelling assumptions, estimated parameter values, and numerical approximations.
- Process improvement. In developing a model, an analyst learns a great deal about the process, and can often identify areas in which the process could be improved.
- Identify and compute desired measures. This is generally the purpose of developing the model in the first place, and is therefore the primary goal.

Once the model has been built, two types of analyses can be performed: steady-state and transient. In a steady-state analysis, one considers the behavior of the system as time tends toward infinity. Predictions of system performance are subsequently based on this limiting behavior. On the other hand, a transient analysis is concerned with the system behavior in the short run. Therefore, predictions must be based on the behavior of the system after a finite amount of time. The next section will discuss some of the known steady-state results for Markov reward processes found in the literature.

2.2 Steady-State Analysis of Markov Reward Processes

An extremely important measure in steady-state analysis is the steady-state probability vector, p, which describes the long-run chance of finding the process in a given state. To find p for a CTMC $\{X(t) : t \geq 0\}$, it is necessary to introduce some notation. First define q_{ij} as the rate at which the CTMC transitions from state i to

state j where $i \neq j$ because a "transition" is defined as moving into a different state. Further, define

$$q_{ii} = -\sum_{j \neq i} q_{ij},$$

and define the square matrix

$$Q = [q_{ij}].$$

The matrix Q is called the infinitesimal generator matrix of $\{X(t) : t \geq 0\}$. Then let

$$p_j = \lim_{t \to \infty} P\{X(t) = j\}$$

denote the long-run probability that the CTMC is in state j. It is well-known [13], [19], that for an irreducible, positive recurrent CTMC, the steady-state distribution is given by the the unique solution to

$$pQ = \mathbf{0} \qquad \sum_{j \in S} p_j = 1, \tag{2.1}$$

where S is the state space.

Analysts can use the steady-state distribution of Equation (2.1) to make predictions about a system's performance in the asymptotic region. For example, let $r_{X(t)}$ denote the reward rate at time t, and let

$$W(t) = \frac{1}{t} \int_0^t X(\tau) d\tau$$

be the time-averaged accumulated reward in the system for $t \geq 0$. Smith, *et al.* [19] show that in the limit as t tends to infinity, the expected values of $r_{X(t)}$ and $W(t)$ are equivalent, and are expressed as

$$\lim_{t \to \infty} E[r_{X(t)}] = \lim_{t \to \infty} E[W(t)] = \sum_i r_i p_i.$$

Although much information can be gleaned from steady-state analysis, it is often more useful to conduct a transient analysis. Obviously, in real applications, many of the measures that analysts seek involve some measure up to a time t. For example, a first passage time is the random time required for a system to first accumulate a specified reward level. Clearly, it makes no sense to use limiting behavior in such a situation.

2.3 Transient Analysis of Markov Reward Processes

A review of the literature on transient analysis of Markov reward processes shows that MRPs have wide applicability in the real world. In this section, several examples from the literature will be provided to show not only the variety of types of systems that can be modelled with a Markov reward process, but also to show the variety of analysis approaches that exist.

The first example comes from [19], in which Smith, *et al.*, present a case study in which they model a multiprocessor system with 16 processors, 16 memories, and a crossbar switch. The "structure-state" of the system at any given time is represented by the triple (i, j, k) indicating the number of operational processors, memories, and switches, respectively. For the system to be functioning, the switch must be operational, and a certain number of processors and memories must be functional. In this paper, they specify that there must be four processors and four memories in operation. Therefore, this system has 169 functioning states and 1 failed state for a total of 170 states. The authors also point out that by altering the types of failure transitions, and the types of repairs, the number of states can grow to as much as 365.

Having established the structure-state process, the authors consider the reward structure for this example. They present three possibilities. The first is a simple availability-based structure in which all operational states are assigned a reward rate

of 1, and the failure state is assigned a reward rate of 0. A second, more accurate, structure is a capacity-based structure in which each operational structure-state $(i, j, 1)$ is assigned reward rate $\min\{i, j\}$, and the failure state is assigned rate 0. The third possible structure, developed by Bhandarkar [4], dictates that the reward rate in an operational state $(i, j, 1)$ is $r_{i,j,1} = m(1-(1-1/m)^L)$, where $L = \min\{i, j\}$ and $m = \max\{i, j\}$. Again, the failure state is assigned rate 0.

Next, the authors examine performability results, based on the 170 state structure-state process, and the contention-based reward structure. With the necessary elements of the Markov reward model defined, they identify and compute four different measures. The first is the computation of availability, $E[X(t)]$, where $X(t)$ is defined as the reward rate at time t, which answers the question "What is the expected performance of the system at time t?" Next, they identify the expected time-averaged accumulated reward over the interval $(0, t)$, which answers the question "What is the time-averaged performance of the system over the interval $(0, t)$?" The third measure they identify is the likelihood of completing a given amount of work in a specified time interval, $\mathcal{Y}^c(x, t)$, which answers the question "What is the probability that x units of work are completed by time t?" The final measure is $\mathcal{W}^c(x, t)$, which answers the question "What is the probability that the reward accumulated in the interval $(0, t)$ is at least xt?"

Because $\mathcal{Y}^c(x, t)$ is a first passage time, it is of particular interest to this thesis. In [19], the authors derive an expression for $\mathcal{Y}^c(x, t)$ in two-dimensional transform space which requires numerical inversion to find values for the cdf of the first passage time for specified values of t. This transform solution is similar to the two-dimensional transform solution reviewed in Chapter 3 of this thesis.

Another example of a real-world computer system modelled as a Markov reward model is given by Kulkarni, *et al.* [12]. The authors present a general model of the completion time of a single job on a computer system whose state changes according to a Markov process. Unlike the previous model, when the state of the system

changes, the service is preempted. The service is then resumed or restarted in the new state at a rate that could possibly be different than the original rate. Thus, this model incorporates three types of service disciplines: preemptive-resume (PRS), in which the job is restarted from the point where it was preempted; preemptive-repeat-identical (PRI), in which the job is restarted from the beginning; and preemptive-repeat-different (PRD), in which the job is restarted with a new work requirement.

In their model, they define $B > 0$ as the amount of work required to complete a job, and $\{Z(t) : t \geq 0\}$ as the continuous-time stochastic process defined on state space $S = \{1, 2, ...\}$. The structure-state of the system is described by this stochastic process. Each state is also identified as either PRS, PRI, or PRD. The reward rate of the system is such that when the system is in state i, the work rate is $r_i \geq 0$. Finally, T is defined as the time needed to complete a job with requirement B. It is the cumulative distribution function of the random variable T that the authors seek.

To find this cdf, Kulkarni, *et al.* [12] take a unique approach which they call "progressive aggregation." This method is described as a 12-step process involving a series of transforms and inversions to ultimately arrive at an expression for the Laplace-Stieltjes transform of F, which is the cdf of T. This expression is a sum of i weighted Laplace-Stieltjes transforms, where i is the number of functional states in the system. The authors demonstrate their technique on a system with four structure-states $\{0, 1, 2, 3\}$, where state 0 is a failure state, state 1 is PRS, state 2 is PRI, and state 3 is PRD. The structure-state process is a CTMC with λ_{ij} denoting the rate of transition from state i to state j, $i \neq j$. The reward rates for states 1, 2, and 3 are r_1, r_2, and r_3, respectively ($r_0 = 0$). Following their 12-step process, they finally arrive at the solution

$$\tilde{F}(s) = \sum_{i=1}^{3} P\{Z(0) = i\}\tilde{F}_i(s),$$

where $\tilde{F}_i(s)$ is the Laplace-Stieltjes transform of $F_i(t) = P\{T \leq t | Z(0) = i\}$.

Although the authors arrive at a solution in one-dimensional transform space, computation could be difficult and extremely time-consuming, particularly as the number of states increases. This problem differs from the one addressed in this thesis; however, it demonstrates that it is possible to accurately model very complex systems using a Markov reward model.

In the literature, there are other examples of computer systems being modelled by Markov reward models, with many researchers finding a solution to the cdf of the first passage time as a two-dimensional Laplace-Stieltjes transform (as seen in Chapter 3). Occasionally, one finds a unique approach such as the previous example. Another exception is the model developed by Nicola, *et.al.* [16] in which the cdf is expressed as a functional equation involving a one-dimensional Laplace-Stieltjes transform. Solving this equation can be difficult, but it provides an alternative to the two-dimensional approach.

Clearly, computer systems are excellent examples of these types of models, but Markov reward models are certainly not limited to computer systems. Kharoufeh [8] provides a different type of example. In his dissertation, he presents a variant of a Markov reward model to analyze a problem from vehicular traffic flow theory. Consider a vehicle travelling along a roadway segment. The various states of the structure-state process are represented by the environmental conditions to which the vehicle is subjected during its sojourn. For example, one state might coincide with rainy conditions, and another might indicate clear conditions. The state of the environment at any given time is governed by a continuous-time Markov chain. The reward rates in this system are the velocities at which the vehicle may travel, and are determined by the state of the system such that when the environmental process is in state i, the vehicle travels at velocity V_i. Finally, the accumulated reward is the total displacement of the vehicle up to a fixed point in time. Kharoufeh [8] seeks to find several measures in this system. The primary measure of interest to this thesis is the unconditional cumulative distribution function of the first passage time $T(x)$,

where $T(x)$ is the random time required to first travel a distance of x. Kharoufeh's solution, which is examined in detail in Chapter 3, is expressed as a two-dimensional transform of the cdf, and must be inverted numerically to obtain results. In addition to the cdf of the first passage time, the author derived the transient and asymptotic moments of the first passage time.

Markov reward models can be advantageous to other alternatives for an assortment of reasons. They are generally less costly in terms of time and money than the alternative methods. Also, the reward rates in a Markov reward process provide modelers with a technique to easily incorporate reliability into their models, making Markov reward processes increasingly more popular. This chapter presented some specific examples found in the literature, as well as the pertinent performance measures. In these examples, it was seen that multiple techniques have been employed to find the cdf of various first passage times for the reward process. Some involved solutions that required numerical inversion of two-dimensional Laplace transforms, and others required extensive calculations to arrive at a solution requiring one-dimensional Laplace inversion. However, the existing literature does not contain a simplified, analytical expression for the Laplace-Stieltjes transform of the cdf in one dimension. The goal of this thesis is to provide a different approach to develop an expression for the first passage time distribution which can be found easily and with very little computation time.

3. Formal Mathematical Model

The primary objective of this chapter is to present a detailed description of Markov reward processes and to provide an explicit result for the first passage time distribution. Additionally, numerical methods for computing approximations to the distributions will be explored.

3.1 Markov Reward Processes

Let $\{Z(t) : t \geq 0\}$ be a continuous-time Markov chain (CTMC) having finite state space $S = \{1, 2, ..., K\}$ for some value $K \in \mathbb{N}$, the set of positive integers. Furthermore, let $Q = [q_{ij}]$, $i, j \in S$, denote the infinitesimal generator matrix for $\{Z(t) : t \geq 0\}$ such that q_{ij} denotes the rate at which the process transitions from state $i \in S$ to $j \in S$, $i \neq j$ and $q_{ii} = -\sum_{i \neq j} q_{ij}$.

The evolution of a Markov reward process can be described as follows. At time 0, the stochastic process, $\{Z(t) : t \geq 0\}$, has initial distribution z_0 and the total accumulated reward (or cost) is zero. Whenever $Z(t)$ assumes a value of $i \in S$, the system accumulates reward at rate $r_i > 0$. The total accumulated reward up to time t is represented by the random variable $R(t)$. Therefore, the process $\{R(t) : t \geq 0\}$ is a continuous-time stochastic process on the continuous space $[0, \infty)$. The reward could be a distance travelled, monetary income, or even a cost such as accumulated damage to a component. For the purposes of this thesis it will be assumed that the reward is non-decreasing. Because the state space is finite, there is also a finite set of reward rates, $\mathcal{R} = \{r_1, r_2, ..., r_K\}$. The accumulated reward in the process can then be expressed as

$$R(t) = R(0) + \int_0^t r_{Z(u)} du,$$

where $R(0)$ is the reward level at time 0, which will be assumed to be 0 throughout. As an example of a Markov reward process, Figure 3.1 demonstrates the relation-

ships between the various random variables in a sample three-state system. In this example, $r_i = 3 - i$.

Now let $p_{ij}(t) = P\{Z(t) = j | Z(0) = i\}$ be the probability that the CTMC is in state j at time t, given that the CTMC was in state i at time 0, where $i, j \in S$. Further, define $P(t) = [p_{ij}(t)]$, the transition probability matrix, which is described by the system of forward Kolmogorov equations

$$\frac{dP(t)}{dt} = P(t)Q, \qquad P_0 \equiv P(0). \tag{3.1}$$

The well-known solution to the initial value problem in (3.1) is

$$P(t) = P_0 \exp{(Qt)}, \tag{3.2}$$

where $\exp{(Qt)}$ is defined by the infinite series

$$\exp{(Qt)} = I + Qt + \frac{Q^2 t^2}{2!} + \frac{Q^3 t^3}{3!} + \ldots \tag{3.3}$$

that is convergent for every choice of Q and t.

Finally, let $R(t)$ be the total reward up to time t, and define

$$T(x) \equiv \inf\{t : R(t) > x\},$$

the random time it takes to first achieve a total reward of $x \in \mathbb{R}^+$. The random time, $T(x)$, is often referred to as the "first passage time" for the process $\{R(t) : t \geq 0\}$. The main objective of this thesis is to explore numerical methods for the evaluation of the cumulative distribution function of $T(x)$.

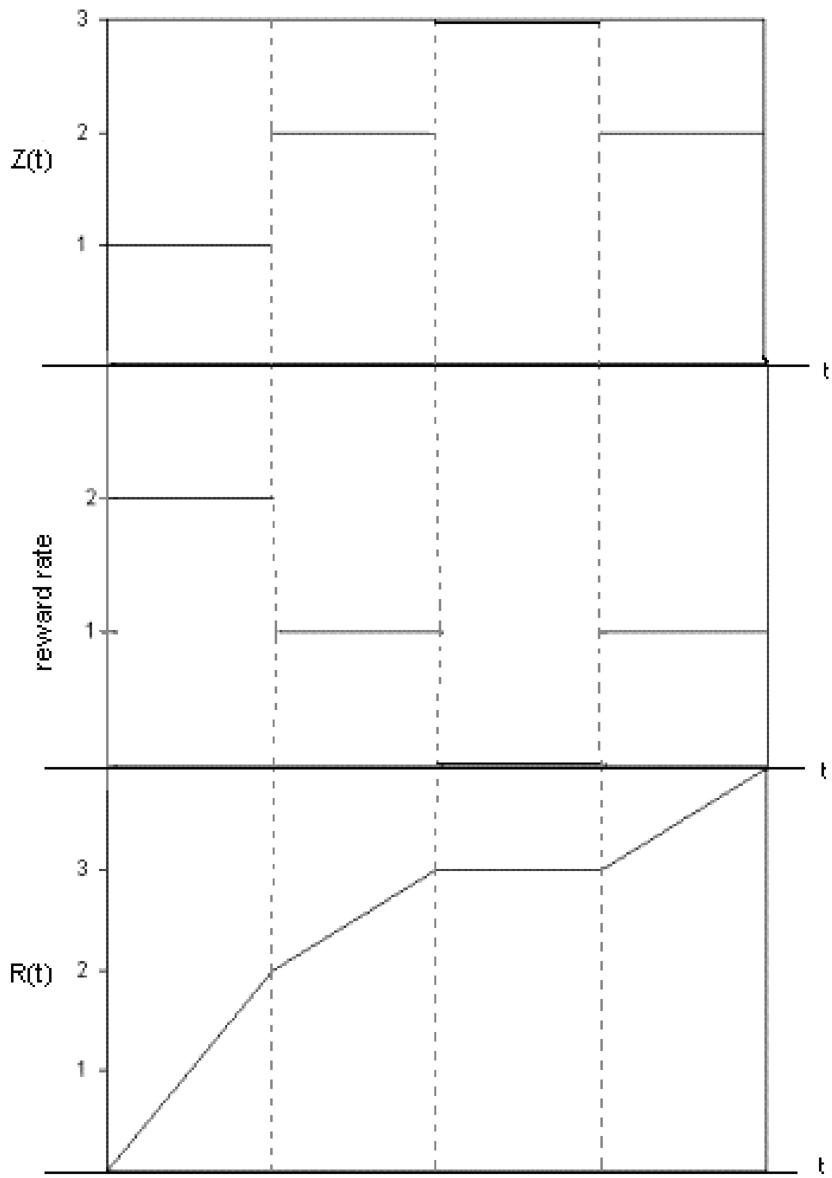

Figure 3.1 Markov reward process: three-state CTMC.

3.2 Distribution of the First Passage Time: Two-Dimensional Approach

In this section, basic probability principles will be used to obtain the distribution of $T(x)$ in two-dimensional transform space. First, it is necessary to define the joint probability distribution

$$H_i(x,t) = P[R(t) \leq x, Z(t) = i]$$

so that by Bayes' Rule,

$$P[R(t) \leq x] = \sum_{i \in S} H_i(x,t).$$

However, it can be shown that under the assumption that $R(t)$ is non-decreasing, the event $\{R(t) \leq x\}$ is equivalent to the event $\{T(x) \geq t\}$. Thus, if $G_x(t)$ is the cumulative distribution function (cdf) of $T(x)$, then

$$G_x(t) = P[T(x) \leq t] = 1 - \sum_{i \in S} H_i(x,t). \tag{3.4}$$

Kharoufeh and Gautam [9] derived the distribution of $T(x)$ by showing that $H_i(x,t)$ satisfies the partial differential equation (PDE)

$$\frac{\partial H_i(x,t)}{\partial t} + \frac{\partial H_i(x,t)}{\partial x} r_i = \sum_{j \in S} q_{ji} H_i(x,t), \quad i \in S \tag{3.5}$$

where $r_i \in \mathcal{R}$. Additionally, set the initial condition $H_i(x,0) = P[Z(0) = i]$. This equation can then be converted to matrix form by letting $H(x,t) = [H_i(x,t)]_{i \in S}$ be the row vector and $V = diag(r_1, r_2, ..., r_K)$. Equation (3.5) then becomes

$$\frac{\partial H(x,t)}{\partial t} + \frac{\partial H(x,t)}{\partial x} V = H(x,t)Q. \tag{3.6}$$

for $t > 0$ and $x > 0$.

Next, define z_0 as the initial distribution of $Z(t)$. Also, define

$$H^*(x, s_2) = \int_0^\infty e^{-s_2 t} H(x,t)dt,$$

the Laplace transform of $H(x,t)$ with respect to t, and

$$\tilde{H}^*(s_1, s_2) = \int_0^\infty e^{-s_1 x} dH^*(x, s_2),$$

the Laplace-Stieltjes transform of $H^*(x, s_2)$ with respect to x. Kharoufeh and Gautam [9] proved that the solution to Equation (3.6) in the transform domain is

$$\tilde{H}^*(s_1, s_2) = z_0(s_1 V + s_2 I - Q)^{-1} \tag{3.7}$$

where s_1 and s_2 are complex variables with $Re(s_1) > 0$ and $Re(s_2) > 0$. Equation (3.7) can be solved numerically using a two-dimensional inversion algorithm. Moorthy [15] and Abate, *et al.* [3] provide two such algorithms, which will be discussed later in this thesis.

When numerically inverting Equation (3.7), one occasionally runs into difficulties due to stability issues relating to the inversion algorithm. This issue will be discussed further in Section 3.4. Additionally, the numerical inversion can often be time consuming in complex problems in which the CTMC has a large number of states, ($K \geq 10$). Because of these problems, it would be beneficial to reduce the problem to a single dimension transform. Numerical inversion in one dimension can be considerably quicker, and does not suffer from the same instability problems.

3.3 Distribution of the First Passage Time: One-Dimensional Approach

The main result of this thesis provides a reduction in the dimensionality of the first passage time distribution of a Markov reward process from two dimensions to

a single dimension and is stated in Theorem 3.1. The result shall be proved in two ways. First via direct methods and second, by inverting the result of Kharoufeh and Gautam [9] with respect to x.

Theorem 3.1 *Let $\tilde{G}_x(s)$ denote the Laplace-Stieltjes transform of the cumulative distribution function of $T(x)$. Then,*

$$\tilde{G}_x(s) = z_0 \exp(V^{-1}[Q - sI]x)\boldsymbol{1}, \tag{3.8}$$

where $\boldsymbol{1} = [1, 1, \cdots, 1]^T$.

3.3.1 Proof of Theorem 3.1: Method One

In this method, the desired result is obtained directly from Equation (3.6),

$$\frac{\partial H(x,t)}{\partial t} + \frac{\partial H(x,t)}{\partial x} V = H(x,t)Q.$$

Taking the Laplace transform of Equation (3.6), with respect to t yields

$$sH^*(x,s) - H(x,0) + \frac{dH^*(x,s)}{dx} V = H^*(x,s)Q, \tag{3.9}$$

since

$$\mathcal{L}\left(\frac{\partial H(x,t)}{\partial t}\right) = sH^*(x,s) - H(x,0).$$

But $H_i(x,0) = P\{R(0) \le x, Z(0) = i\} = P\{Z(0) = i\}$ and therefore $H(x,0) = z_0$, where z_0 is the initial distribution vector of $Z(t)$. Equation (3.9) then becomes

$$sH^*(x,s) - z_0 + \frac{dH^*(x,s)}{dx} V = H^*(x,s)Q. \tag{3.10}$$

Rearranging terms leaves a linear ordinary differential equation (ODE) system with constant coefficients,

$$\frac{dH^*(x,s)}{dx} + H^*(x,s)[sI - Q]V^{-1} = z_0 V^{-1}. \tag{3.11}$$

For this ODE, the necessary integrating factor is

$$\mu(x) = \exp\left(\int [sI - Q]V^{-1} dx\right) = \exp([sI - Q]V^{-1}x),$$

since $[sI - Q]V^{-1}$ is a constant matrix with respect to x. Right multiplying both sides of Equation (3.11) by $\mu(x)$ yields

$$(H^*(x,s)\exp([sI - Q]V^{-1}x))' = z_0 V^{-1}\exp([sI - Q]V^{-1}x). \tag{3.12}$$

Integrating both sides of Equation (3.12) with respect to x produces

$$H^*(x,s)\exp([sI - Q]V^{-1}x) = z_0 V^{-1}(V[sI - Q]^{-1})\exp([sI - Q]V^{-1}x) + C, \tag{3.13}$$

where C is the constant row vector of integration. Simplifying and applying the initial condition $H^*(0,s) = 0$ shows that $C = -z_0[sI - Q]^{-1}$. This results in

$$H^*(x,s) = z_0[sI - Q]^{-1} - z_0[sI - Q]^{-1}\exp([Q - sI]V^{-1}x), \tag{3.14}$$

which can be shown to be equivalent to

$$H^*(x,s) = z_0[sI - Q]^{-1} - z_0\exp(V^{-1}[Q - sI]x)[sI - Q]^{-1}. \tag{3.15}$$

Define the Laplace-Stieltjes transform of the distribution of the first passage time as

$$\mathcal{L}[G_x](s) = \tilde{G}_x(s) = \int_0^\infty e^{-st} dG_x(t),$$

which is

$$\tilde{G}_x(s) = 1 - H^*(x,s)s\mathbf{1}, \tag{3.16}$$

where $\mathbf{1}$ is a column vector of ones. Combining Equations (3.14) and (3.16) produces

$$\tilde{G}_x(s) = 1 - (z_0 - z_0 \exp(V^{-1}[Q - sI]x))[sI - Q]^{-1}s\mathbf{1}. \tag{3.17}$$

This reduces to

$$\tilde{G}_x(s) = 1 - (z_0 - z_0 \exp(V^{-1}[Q - sI]x)) \left[I - \frac{Q}{s}\right]^{-1} \mathbf{1}. \tag{3.18}$$

Using the Neumann expansion [17] $(I - A)^{-1} = I + A + A^2 + A^3 + \cdots$, applied to $A = Q/s$, where $\|A\| = \max_i \sum_j |a_{ij}| < 1$, Equation (3.18) becomes

$$\tilde{G}_x(s) = 1 - (1 - z_0 \exp(V^{-1}[Q - sI]x)\mathbf{1}), \tag{3.19}$$

since $Q\mathbf{1} = \mathbf{0}$ and $z_0\mathbf{1} = 1$ for an irreducible, finite-state CTMC. This gives the desired result

$$\tilde{G}_x(s) = z_0 \exp(V^{-1}[Q - sI]x)\mathbf{1}. \tag{3.20}$$

3.3.2 Proof of Theorem 3.1: Method Two

In the second method, the desired one-dimensional result is obtained from the two-dimensional result in Equation (3.7).

From Equation (3.16),

$$\tilde{G}_x(s) = 1 - H^*(x,s)s\mathbf{1}.$$

Taking the Laplace-Stieltjes transform of both sides with respect to x, and combining the result with Equation (3.7) yields

$$\hat{G}(w) = 1 - \tilde{H}^*(w,s)s\mathbf{1} = 1 - z_0(wV + sI - Q)^{-1}\mathbf{1}s,$$

where $\hat{G}(w)$ represents the double Laplace-Stieltjes transform of $G(x)$. Since Laplace transforms are easier to invert, the above equation is written in terms of the Laplace transform as

$$w\tilde{G}^*(w) = 1 - z_0(wV + sI - Q)^{-1}\mathbf{1}s,$$

because $\hat{G}$ is related to $\tilde{G}^*$ by the equation [13]

$$\hat{G}(w) = w\tilde{G}^*(w).$$

Rearranging terms results in

$$w\tilde{G}^*(w) = 1 - \frac{z_0}{w}\left[I - \frac{V^{-1}(Q - sI)}{w}\right]^{-1} V^{-1}\mathbf{1}s.$$

Again using the geometric expansion $(I - A)^{-1} = I + A + A^2 + A^3 + \cdots$, for $\|A\| < 1$, and dividing by w gives

$$\tilde{G}^*(w) = \frac{1}{w} - \frac{z_0}{w^2}\left[I + \sum_{n=1}^{\infty}\left(\frac{V^{-1}(Q - sI)}{w}\right)^n\right] V^{-1}\mathbf{1}s,$$

for $\|V^{-1}(Q - sI)\| < 1$. In order to facilitate inversion, the above equation is rewritten as

$$\tilde{G}^*(w) = \frac{1}{w} - z_0\left[\frac{I}{w^2} + \sum_{n=1}^{\infty}\left(V^{-1}(Q - sI)\right)^n \frac{1}{w^{n+2}}\right] V^{-1}\mathbf{1}s.$$

Since the inverse Laplace transform is uniformly continuous, the transform of the infinite sum is the infinite sum of the transforms. That is,

$$\mathcal{L}\left[\sum_{n=1}^{\infty}(V^{-1}(Q-sI))^n\frac{1}{w^{n+2}}\right]=\sum_{n=1}^{\infty}(V^{-1}(Q-sI))^n\mathcal{L}\left[\frac{1}{w^{n+2}}\right].$$

The terms of this series can now be inverted, resulting in

$$\tilde{G}_x(s)=1-z_0\left[Ix+\sum_{n=1}^{\infty}\left(V^{-1}(Q-sI)\right)^n\frac{x^{n+1}}{(n+1)!}\right]V^{-1}\mathbf{1}s.$$

After some algebra, the above equation can be written as

$$\tilde{G}_x(s)=1-z_0\left[xV^{-1}(Q-sI)+\sum_{n=1}^{\infty}\frac{\left(V^{-1}(Q-sI)x\right)^{n+1}}{(n+1)!}\right](Q-sI)^{-1}\mathbf{1}s. \quad (3.21)$$

Since the exponential of a matrix is defined as $\exp(A)=\sum_{j=0}^{\infty}A^j/j!$, Equation (3.21) can be written as

$$\tilde{G}_x(s)=1-z_0\left[-I+\exp(V^{-1}(Q-sI)x)\right](Q-sI)^{-1}\mathbf{1}s.$$

Once again, expanding $(Q-sI)^{-1}$ by the Neumann expansion yields

$$\tilde{G}_x(s)=1+z_0\left[-I+\exp(V^{-1}(Q-sI)x)\right]\left(I+\frac{Q}{s}+\frac{Q^2}{s^2}+\frac{Q^3}{s^3}+\cdots\right)\mathbf{1}.$$

However, because $Q\mathbf{1}=\mathbf{0}$ and $z_0\mathbf{1}=1$, the result is

$$\tilde{G}_x(s)=z_0\exp(V^{-1}(Q-sI)x)\mathbf{1}. \quad (3.22)$$

3.4 Numerical Inversion

Once a solution is obtained in the transform domain, the problem of inverting that solution remains. In certain cases, an exact expression for the inverse transform

can be obtained. For example, if the transform solution is a vector of rational functions in all the complex variables, partial fraction decomposition can be employed to obtain a form which facilitates analytic inversion via the inverse Laplace tables. However, this can be a difficult task. Therefore, obtaining approximations by numerical inversion is generally preferred. At this point, there will be a brief review of numerical inversion of Laplace transforms in both one and two dimensions.

3.4.1 Numerical Inversion in One Dimension

Recall that the Laplace transform of a function $f(t)$ and its inverse transform are, respectively,

$$f^*(s) = \int_0^\infty e^{-st} f(t)dt, \tag{3.23}$$

and

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} f^*(s)ds, \tag{3.24}$$

where $a > 0$ is arbitrary, but must be greater than the real parts of all the singularities of $f^*(s)$. In this thesis, it is assumed that $f(t)$ is a real-valued function, so that Equation (3.23) and Equation (3.24) can be replaced by

$$\mathrm{Re}\{f^*(s)\} = \int_0^\infty e^{-at} f(t)\cos(wt)dt, \tag{3.25}$$

and

$$f(t) = \frac{2e^{at}}{\pi} \int_0^\infty Re\{f^*(s)\}\cos(wt)dw, \tag{3.26}$$

where $s = a + iw$. Solving Equation (3.26) is generally not an easy task. Therefore, numerical techniques for approximating $f(t)$ are used.

One of the most common techniques for approximating the integral of Equation (3.26) is to exploit the relationship between the Laplace transform and the finite Fourier cosine transform. The paper by Dubner and Abate [6] is widely regarded as the seminal paper on this technique, and will be summarized in this section.

Consider a real function, $h(t)$ such that $h(t) = 0$ for $t < 0$, that has been broken into sections, each of length T. That is, there are then an infinite number of intervals $(nT, (n+1)T)$ for $n = 0, 1, 2, \cdots$. Each of these is reflected through its boundary, constructing an infinite number of even periodic functions $g_n(t)$, each with period $2T$. Therefore,

$$g_n(t) = \begin{cases} h(t) & : \quad nT \leq t \leq (n+1)T \\ h(2nT - t) & : \quad (n-1)T \leq t \leq nT. \end{cases} \tag{3.27}$$

Each $g_n(t)$ is then rewritten so that the functions are defined on $(-T, T)$. For $n = 0, 2, 4, \cdots$,

$$g_n(t) = \begin{cases} h(nT + t) & : \quad 0 \leq t \leq T \\ h(nT - t) & : \quad -T \leq t \leq 0, \end{cases} \tag{3.28}$$

and for $n = 1, 3, 5, \cdots$,

$$g_n(t) = \begin{cases} h((n+1)T - t) & : \quad 0 \leq t \leq T \\ h((n+1)T + t) & : \quad -T \leq t \leq 0, \end{cases} \tag{3.29}$$

Taking the Fourier representation of each $g_n(t)$, and summing them yields

$$\sum_{n=0}^{\infty} g_n(t) = \frac{2}{T} \left[\frac{A(w_0)}{2} + \sum_{k=1}^{\infty} A(w_k) \cos\left(\frac{k\pi}{T} t\right) \right], \tag{3.30}$$

where

$$A(w_k) = \int_0^{\infty} h(t) \cos\left(\frac{k\pi}{T} t\right) dt. \tag{3.31}$$

$A(w_k)$ is a Fourier cosine transform, but if an attenuation factor is introduced by letting

$$h(t) = e^{-at} f(t), \tag{3.32}$$

it becomes the Laplace transform of the real function $f(t)$ with transform variable $s = a + i(k\pi/T)$. Therefore, $A(w_k) = \mathrm{Re}\{f^*(s)\}$. Multiplying both sides of Equation

(3.30) by the attenuation factor e^{at} yields

$$\sum_{n=0}^{\infty} e^{at} g_n(t) = \frac{2e^{at}}{T}\left[\frac{1}{2}\mathrm{Re}\{f^*(a)\} + \sum_{k=1}^{\infty}\mathrm{Re}\left\{f^*\left(a+\frac{k\pi i}{T}\right)\right\}\cos\frac{k\pi}{T}t\right]. \tag{3.33}$$

Dubner and Abate [6] conclude that for any t such that $0 \leq t \leq T/2$, the inverse Laplace transform can be approximated to any desired accuracy by

$$f(t) \approx \sum_{n=0}^{\infty} e^{at} g_n(t) = \frac{2e^{at}}{T}\left[\frac{1}{2}\mathrm{Re}\{f^*(a)\} + \sum_{k=1}^{\infty}\mathrm{Re}\left\{f^*\left(a+\frac{k\pi i}{T}\right)\right\}\cos\frac{k\pi}{T}t\right]. \tag{3.34}$$

Examination of Equation (3.34) reveals that the approximation formula is simply an application of the trapezoidal rule [14] to Equation (3.26). All one-dimensional inversion results in this thesis will be obtained using algorithms coded in MATLAB®, based on Equation (3.34).

There are, however, other techniques to perform one-dimensional inversion. Abate, *et al.* [2] develop one such technique in which the desired function can be approximated as a weighted sum of Laguerre functions. With some types of problems, this method performs better than the Fourier series method, but in others it performs worse [2]. In fact, testing by Davies and Martin [5] on 16 test problems showed that the Laguerre method gave poor results on 6 of those test problems. In general, the Laguerre method is considered unsuitable with two types of problems. It has difficulties with problems in which the Laguerre functions fail to converge geometrically, and problems in which there is geometric convergence of the Laguerre functions, but in which large t values are considered.

3.4.2 Numerical Inversion in Two Dimensions

Many types of problems encountered in the real world involve Laplace transformations in two dimensions. In this case, Equations (3.25) and (3.26) extend

naturally to

$$f^*(s_1, s_2) = \int_0^\infty \int_0^\infty e^{-s_1 t_1 - s_2 t_2} f(t_1, t_2) dt_1 dt_2, \tag{3.35}$$

and

$$f(t_1, t_2) = \frac{1}{(2\pi i)^2} \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} e^{s_1 t_1 + s_2 t_2} f^*(s_1, s_2) ds_1 ds_2, \tag{3.36}$$

where $f(t_1, t_2)$ is a real-valued function of t_1 and t_2, $f(t_1, t_2) = 0$ for t_1 or $t_2 < 0$, and $|f(t_1, t_2)| \leq M e^{\alpha_1 t_1 + \alpha_2 t_2}$, where α_1 and α_2 are real numbers and M is a positive constant. It is also assumed that $c_1 > \alpha_1$ and $c_2 > \alpha_2$.

Moorthy [15] provides a technique to approximate $f(t_1, t_2)$ by applying the technique of Dubner and Abate [6] in two dimensions. Consequently, $f(t_1, t_2)$ can be approximated by

$$\begin{aligned}
f_N(t_1, t_2) &= \frac{1}{2T^2} \left\{ \frac{1}{2} f^*(c_1, c_2) + \sum_{m=1}^{N} \left[\mathrm{Re} \left\{ f^* \left(c_1, c_2 + \frac{im\pi}{T} \right) \right\} \cos \left(\frac{m\pi t_2}{T} \right) \right. \right. \\
&- \left. \mathrm{Im} \left\{ f^* \left(c_1, c_2 + \frac{im\pi}{T} \right) \right\} \sin \left(\frac{m\pi t_2}{T} \right) \right] \\
&+ \sum_{n=1}^{N} \left[\mathrm{Re} \left\{ f^* \left(c_1 + \frac{in\pi}{T}, c_2 \right) \right\} \cos \left(\frac{n\pi t_1}{T} \right) \right. \\
&- \left. \mathrm{Im} \left\{ f^* \left(c_1 + \frac{in\pi}{T}, c_2 \right) \right\} \sin \left(\frac{n\pi t_1}{T} \right) \right] \\
&+ \sum_{m=1}^{N} \sum_{n=1}^{N} \left[\mathrm{Re} \left\{ f^* \left(c_1 + \frac{in\pi}{T}, c_2 + \frac{im\pi}{T} \right) \right\} \cos \left(\frac{n\pi t_1}{T} + \frac{m\pi t_2}{T} \right) \right. \\
&+ \mathrm{Re} \left\{ f^* \left(c_1 + \frac{in\pi}{T}, c_2 - \frac{im\pi}{T} \right) \right\} \cos \left(\frac{n\pi t_1}{T} - \frac{m\pi t_2}{T} \right) \\
&- \mathrm{Im} \left\{ f^* \left(c_1 + \frac{in\pi}{T}, c_2 + \frac{im\pi}{T} \right) \right\} \sin \left(\frac{n\pi t_1}{T} + \frac{m\pi t_2}{T} \right) \\
&- \left. \left. \mathrm{Im} \left\{ f^* \left(c_1 + \frac{in\pi}{T}, c_2 - \frac{im\pi}{T} \right) \right\} \sin \left(\frac{n\pi t_1}{T} + \frac{m\pi t_2}{T} \right) \right] \right\}.
\end{aligned} \tag{3.37}$$

To use the above approximation, it is necessary to select values for the parameters c_1, c_2, and N. Selecting an appropriate value for N helps to control truncation error. Generally, N is found by determining an N for which the difference between

f_{N+1} and $f_{N+N/4}$ is negligible. The variables c_1 and c_2 can be assigned arbitrarily, provided they meet the previously mentioned restriction. However, for many functions, it is not easy to determine the values of α_1 and α_2, making the assignment of c_1 and c_2 difficult as well.

As in one-dimensional inversion, there are other alternatives to the Fourier method. One such alternative was proposed by Abate, *et al.*[3], in which the Laguerre method is extended to multiple dimensions. According to the authors, the Laguerre method provides good results for well-behaved functions (*i.e.*, the function and its derivatives are sufficiently smooth), but that the Fourier series method proved to be more robust. Like the Laguerre method in one dimension, the multidimensional extension performs poorly in problems with large t values, and problems with non-geometric convergence of the Laguerre functions.

In this chapter, a detailed description of Markov reward processes was presented. Several results from a transient analysis of such processes were examined, including the concept of a first passage time. Building on the existing literature on the subject, this chapter developed a simplified, analytical expression for the Laplace-Stieltjes transform of the cdf of the first passage time in one dimension. It remains to demonstrate this analytical result by applying it to real-world operational problems, and comparing the results with numerical results obtained by previously established techniques. That will be the focus of the next chapter.

4. Numerical Examples

In this chapter, applications of Markov reward processes will be considered. The examples demonstrate the usefulness of the techniques presented in this thesis and involve problems from a variety of real-world scenarios. In some cases, the results from one-dimensional inversion will be directly compared to results from two-dimensional inversion.

4.1 Problems in Reliability Theory

In this section, the utility of Equation (3.20) will be demonstrated through a problem involving the propagation of a crack in metallic materials. The problem will be shown to be a Markov reward process and the technique developed in this thesis will be used to find numerical solutions for the desired distribution. These solutions will be compared to solutions obtained in previous work by Kharoufeh [11], obtained through two-dimensional methods, as well as with simulation results.

Consider a new metallic component which has just been placed in service. Initially, there are no cracks in the component, but in time, normal wear and fatigue cause the initiation of a crack. Continued operation causes the crack to grow. The rate at which this crack grows is completely determined by the system's random environment, which consists of two distinct states. State 1 causes the crack to grow with rate r_1 and state 2 causes it to grow with rate r_2. Furthermore, it is assumed the this random environment can be characterized by a continuous-time Markov chain, $\{Z(t) : t \geq 0\}$, which alternates between the two states, $S = \{1, 2\}$, and has infinitesimal generator matrix

$$Q = \begin{bmatrix} -b & b \\ a & -a \end{bmatrix},$$

where b is the rate at which the environment transitions from state 1 to state 2, and a is the rate at which the environment transitions from state 2 to state 1.

This problem can be considered as a Markov reward process. First, let $X(t)$ denote the length of the crack at time t. This is the reward (cost) in the system, which was previously denoted $R(t)$ in Chapter 3. The set of rates at which the crack grows, $\{r_1, r_2\}$, represents the set of reward rates, $\mathcal{R}$, from Section 3.1. Finally, let R_D be the diagonal matrix of wear rates, which coincides with the V matrix from before. In this problem, $T(x)$ is the random time required for the crack to first reach a certain fixed length $x \in \mathbb{R}^+$ (e.g., the time required for the component to fail). It is the cumulative distribution function of the random variable $T(x)$ that is desired.

For the specific problem solved by Kharoufeh [11], the parameter values are as follows. The wear rates are $r_1 = 1.0833$ and $r_2 = 0.250$. The generator matrix values are $a = b = 25/3$ and the crack length threshold is set at $x = 1.0$. Additionally, it is assumed, with probability 1, the system begins in state 1. Therefore, the necessary matrices are

$$Q = \begin{bmatrix} -25/3 & 25/3 \\ 25/3 & -25/3 \end{bmatrix},$$

$$R_D = \begin{bmatrix} 1.0833 & 0 \\ 0 & 0.250 \end{bmatrix},$$

and

$$z_0 = [1 \quad 0].$$

Next, Equation (3.20) can be used to calculate a solution in the transform space. Using the given matrices, that solution is

$$\tilde{G}_x(s) =$$
$$[1 \quad 0] \exp\left(\begin{bmatrix} 1.0833 & 0 \\ 0 & 0.250 \end{bmatrix}^{-1} \left[\begin{bmatrix} -25/3 & 25/3 \\ 25/3 & -25/3 \end{bmatrix} - s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] 1 \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Table 4.1 Cumulative probability values for a 2-state reliability problem.

x	t	*2-D Inversion*	*1-D Inversion*	*Simulated*
1.0	1.197	0.1259	0.1270	0.1288
1.0	1.288	0.2373	0.2379	0.2430
1.0	1.379	0.3720	0.3719	0.3704
1.0	1.470	0.5128	0.5122	0.5140
1.0	1.562	0.6437	0.6442	0.6484
1.0	1.653	0.7539	0.7543	0.7501
1.0	1.744	0.8396	0.8397	0.8360
1.0	1.835	0.9010	0.9009	0.9002
1.0	1.926	0.9421	0.9419	0.9424
1.0	2.018	0.9677	0.9679	0.9698
1.0	2.109	0.9830	0.9830	0.9829
1.0	2.200	0.9915	0.9915	0.9914
1.0	2.291	0.9958	0.9959	0.9957
1.0	2.382	0.9982	0.9981	0.9980
1.0	2.474	0.9991	0.9992	0.9994
1.0	2.565	0.9995	0.9997	0.9996
1.0	2.656	0.9999	0.9999	0.9996
1.0	2.747	0.9999	1.0000	1.0000

The one-dimensional Laplace transform inversion algorithm of Abate and Whitt [1] was used to compute the analytical cumulative distribution values at various points in time. These results are shown in Table 4.1, along with empirical results from Monte-Carlo simulation. A comparison of the results from the one- and two-dimensional methods shows that the techniques provide similar results, which compare well with the benchmark simulation results. This favorable comparison is easily seen in Figure 4.1 which plots the distribution for each method.

4.2 Problems in Vehicular Traffic Flow Theory

In this section, two numerical examples will be given in which a first passage time cumulative distribution is obtained for a vehicle whose velocity is controlled by a CTMC. The first example will involve a two-state CTMC, and the second example will involve a five-state CTMC. Both examples were used in Kharoufeh's

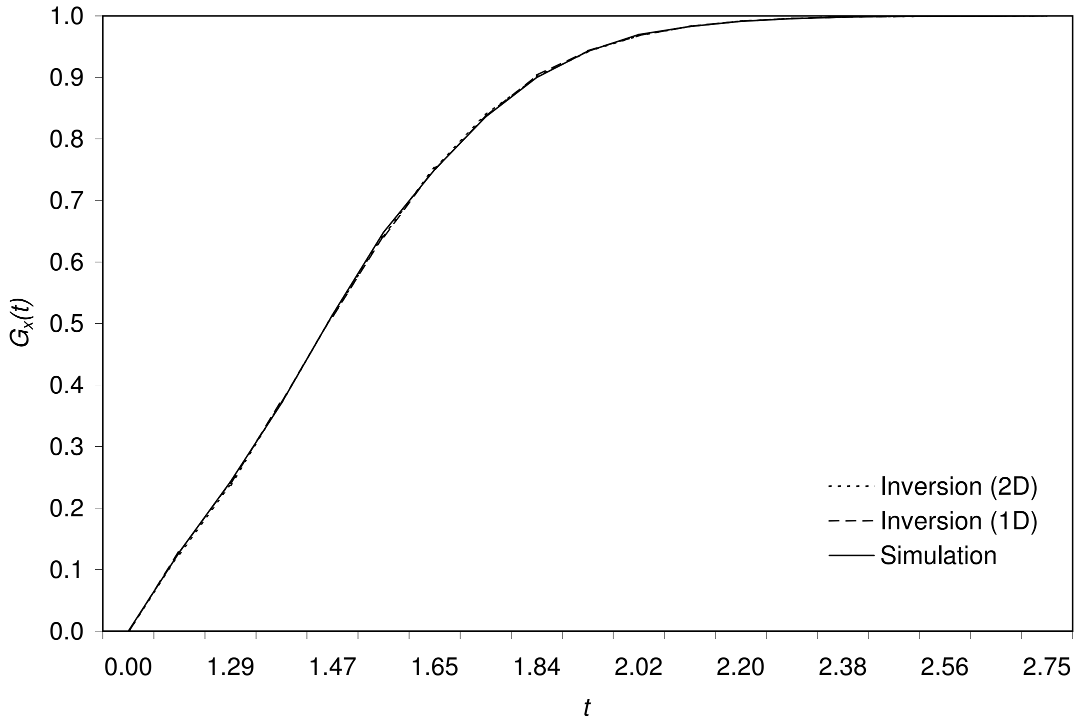

Figure 4.1 Cumulative distribution of $T(x)$.

[8] derivation of two-dimensional results, and will provide a opportunity to compare one- and two-dimensional results.

4.2.1 Two-State CTMC

In this example, a vehicle is travelling along a straight line, and may assume either of two velocities, $V_1 = 50$ mph, or $V_2 = 30$ mph. The particular velocity assumed at any given time is controlled by a two-state environmental process, $\{Z(t) : t \geq 0\}$, having state space $S = \{1, 2\}$. One can think of the two states in terms of conditions which would normally affect a vehicle's velocity. For example, perhaps state 1 equates to a clear day, while state 2 represents rain or snow. Therefore, when $Z(t)$ is in state i, the vehicle has velocity V_i, $i = 1, 2$. The amount of time the system spends in state 1 is exponentially distributed with rate β, and the amount of time spent in state 2 is exponentially distributed with rate α. For this problem,

$\alpha = \beta = 500hr^{-1}$. Additionally, it is arbitrarily assumed that the system starts in state 1 at time 0, so that $P\{Z(0) = 1\} = 1$.

Clearly the problem reduces to the basic Markov reward process described in Chapter 3. In this case, the accumulated reward, $R(t)$, is the total distance travelled by the vehicle up to time t. while the set of reward rates, $\mathcal{R}$, is the set of velocities which the vehicle may assume. Finally, the environmental process in this example coincides with the stochastic process, $\{Z(t) : t \geq 0\}$, in Section 3.1.

The first step in obtaining the cumulative distribution of $R(t)$ is to identify the necessary matrices. The infinitesimal generator matrix for the CTMC in this example is

$$Q = \begin{bmatrix} -\beta & \beta \\ \alpha & -\alpha \end{bmatrix} = \begin{bmatrix} -500 & 500 \\ 500 & -500 \end{bmatrix}.$$

The V matrix is

$$V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} = \begin{bmatrix} 50 & 0 \\ 0 & 30 \end{bmatrix}.$$

Because this problem seeks cdf values for values of t in minutes rather than hours, each of these matrices is scaled by 60. Finally, because the problem assumes (with probability 1) that the system starts in state 1 at time 0, the initial distribution of the governing process is

$$z_0 = [1 \quad 0].$$

Equation (3.20),

$$\tilde{G}_x(s) = z_0 \exp(V^{-1}[Q - sI]x)\mathbf{1},$$

can now be applied to solve for the cumulative distribution at any value of t for a particular value of x. For example, if is desired to find the cumulative distribution of the random time it takes to travel a total distance of 1 mile (i.e., $P\{T(1) \leq t\}$),

Table 4.2 Cumulative probability values for a 2-state transportation problem.

x (mi)	t (min)	*1-D Inversion*	*Simulated*
1.0	1.25	0.0036	0.0033
1.0	1.30	0.0243	0.0242
1.0	1.35	0.0850	0.0853
1.0	1.40	0.2019	0.2006
1.0	1.45	0.3691	0.3682
1.0	1.50	0.5570	0.5572
1.0	1.55	0.7278	0.7283
1.0	1.60	0.8557	0.8559
1.0	1.65	0.9348	0.9347
1.0	1.70	0.9754	0.9762
1.0	1.75	0.9925	0.9927
1.0	1.80	0.9983	0.9983
1.0	1.85	0.9998	0.9996
1.0	1.90	1.0000	1.0000

the above equation becomes

$$\tilde{G}_x(s) =$$
$$\begin{bmatrix} 1 & 0 \end{bmatrix} \exp\left(\begin{bmatrix} 50 & 0 \\ 0 & 30 \end{bmatrix}^{-1} \left[\begin{bmatrix} -500 & 500 \\ 500 & -500 \end{bmatrix} - s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] 1 \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The above equation can be solved, and subsequently inverted using an algorithm coded into the MATLAB® software. The functions in the code which calculate the matrix exponentiation in the above equation utilize the Padé approximation approach, which is covered in great depth in [20].

The numerical results for various values of t are given in Table 4.2 along with results from Monte-Carlo simulation. Comparing the results from one-dimensional inversion with the simulation results shows that the one-dimensional method gives acceptable results for this problem.

4.2.2 *Five-State CTMC*

In this example, a problem similar to that of Section 4.2.1 will be examined. In this case, the environmental process, $\{Z(t) : t \geq 0\}$, is a five-state CTMC with state space $S = \{1, 2, 3, 4, 5\}$. When the environment process is in state i, the vehicle assumes a velocity of $V_i = 75/i$ for $i \in S$. As in the previous problem, it is assumed that, with probability 1, the system begins in state 1 at time 0.

For this problem, the off-diagonal entries of the infinitesimal generator matrix were assigned randomly, based on a uniform distribution on the interval (200,400) hr^{-1}. The resulting generator matrix is

$$Q = \begin{bmatrix} -919.75 & 206.91 & 264.85 & 238.67 & 209.32 \\ 223.01 & -971.71 & 301.98 & 232.73 & 213.98 \\ 343.04 & 277.78 & -1283.57 & 392.72 & 270.03 \\ 353.91 & 232.27 & 213.69 & -1059.47 & 259.59 \\ 370.92 & 200.89 & 216.80 & 225.60 & -1014.21 \end{bmatrix}.$$

The other required matrices are

$$V = \begin{bmatrix} 75/1 & 0 & 0 & 0 & 0 \\ 0 & 75/2 & 0 & 0 & 0 \\ 0 & 0 & 75/3 & 0 & 0 \\ 0 & 0 & 0 & 75/4 & 0 \\ 0 & 0 & 0 & 0 & 75/5 \end{bmatrix},$$

and

$$z_0 = [1 \quad 0 \quad 0 \quad 0 \quad 0].$$

As before, the matrices are scaled by 60, the distance is set to $x = 1.0$ mile, and $P\{T(x) \leq t\}$ was computed for several values of t. The results are shown in Table 4.3. Again, it is evident that both the one- and two-dimensional techniques provide

Table 4.3 Cumulative probability values for a 5-state transportation problem.

x (mi)	t (min)	*2-D Inversion*	*1-D Inversion*	*Simulated*
1.0	1.25	0.0786	0.0806	0.0777
1.0	1.47	0.3335	0.3311	0.3352
1.0	1.70	0.6859	0.6922	0.6865
1.0	1.92	0.9141	0.9144	0.9136
1.0	2.14	0.9873	0.9868	0.9872
1.0	2.37	0.9991	0.9990	0.9991
1.0	2.59	1.0000	0.9999	0.9999
1.0	2.81	1.0000	0.9999	1.0000

acceptable results. However, the amount of time saved by using the one-dimensional technique grows dramatically as the number of states in the CTMC increases. There is no noticeable difference in computation time in the one-dimensional inversion process between the two-state and five-state problems, whereas with the two-dimensional inversion process, the time required to find a solution with 5 states is considerably greater than with 2 states. In fact, it was noted that the time required to calculate the cumulative probability at one point in time in the 5-state problem is nearly 30 times greater than the time required in the two-state problem.

4.3 A Problem from the Theory of Queues

In this section, a numerical problem will be considered involving a fluid queue. For this particular problem, the rate of fluid flow into an infinite-capacity buffer (queue) is stochastic and time-variant.

Consider a sewage treatment facility in which the rate at which sewage arrives is dependent on the state of the system. These states might represent various levels of disrepair in the system and/or blockages in the incoming pipes. Assume that there are 20 possible states in this system. In this problem, $R(t)$ denotes to the cumulative amount of sewage flow into the queue (the treatment facility) up to time t. The state of the environment at time t is modelled by the CTMC $\{Z(t) : t \geq 0\}$ with state space $S = \{1, 2, \cdots, 20\}$. The rate at which the sewage arrives is governed

Table 4.4 Cumulative probability values for a fluid queueing problem.

x	t	*1-D Inversion*	*Simulated*
1.0	0.040	0.0091	0.0087
1.0	0.045	0.0565	0.0584
1.0	0.050	0.2090	0.2083
1.0	0.055	0.4875	0.4894
1.0	0.060	0.7752	0.7796
1.0	0.065	0.9402	0.9417
1.0	0.070	0.9912	0.9910
1.0	0.075	0.9993	0.9991
1.0	0.080	1.0000	0.9999

by this CTMC, and the set of possible rates is $\mathcal{R} = \{r_1, r_2, \cdots, r_{20}\}$ such that when $Z(t) = i$, the sewage arrives at rate r_i. Modelling the system as a Markov reward process allows the time-variant and stochastic nature of the arrival process to be incorporated into the analysis. Suppose an analyst is interested in the amount of time required to accumulate 1 unit of total flow into the facility without regard to its ultimate destination. He or she may desire to find the cumulative distribution of $T(1)$, the random time required to first accumulate 1 unit of fluid.

Again, the first step in finding a solution is to identify the necessary matrices. The Q matrix is the infinitesimal generator matrix for $\{Z(t) : t \geq 0\}$. For this problem, the off-diagonal entries of the generator matrix were uniformly distributed on the interval (100, 300). When the system is in state i, the rate at which sewage arrives is $r_i = 100/i$. Finally, it is assumed, with probability 1, that the system begins in state 1. Due to the size of these matrices, they are not displayed in this thesis; however the cumulative probability results, along with Monte-Carlo simulation results, are shown in Table 4.4.

The major benefit of the one-dimensional technique becomes more and more apparent as the dimensionality of the Z process increases. Calculating the distribution results in Table 4.4 via one-dimensional inversion required approximately two seconds on a Pentium III 1000 MHz laptop, whereas Monte-Carlo simulation of the

Table 4.5 Approximate Computation Times for Numerical Examples

States	*1-D Inversion*	*2-D Inversion*
2	2 seconds	15 seconds
5	2 seconds	7 minutes
20	2 seconds	3 hours

problem on the same laptop requires approximately 20 minutes and two-dimensional inversion would require about 3 hours. Obviously, as the problem size increases, these time discrepancies are expected to be even more pronounced.

The examples in this chapter clearly demonstrate the usefulness of having a closed-form expression for the Laplace-Stieltjes transform of the cdf of the first passage time in one dimension. The numerical results from this technique are essentially equivalent to those obtained from the previous two-dimensional technique (based on comparison with benchmark simulation results). However, the one-dimensional method provides tremendous time savings which are illustrated in table 4.5. The order of magnitude of this time savings grows as the number of states in the CTMC grows, making it a particularly valuable technique for large problems.

5. Conclusions and Future Research

This thesis presented techniques for analyzing the cumulative distribution function of the first passage time of a Markov reward process. Specifically, an exact distribution of first passage times in one-dimensional transform space was developed. Additionally, real-world applications of these types of stochastic, time-varying systems were examined, not only to demonstrate the utility of this type of modelling, but also to provide numerical examples on which this technique could be tested.

The first step in the study was to formally define a Markov reward process (MRP), as well as the concept of a first passage time. The MRP was presented as a system consisting of a finite-state Markov process that accumulates reward over time. The rate at which this system accumulates reward (or cost) is governed directly by the state of the Markov process. The first passage time was defined as the random time required to first accumulate a given reward level. It was the cumulative distribution of this random time that was of particular interest to this study. A review of the existing literature on the subject revealed that solutions for this distribution are typically found in two-dimensional transform space. Numerical inversion is usually employed to produce approximations for cumulative distribution values. This thesis demonstrated this technique by reviewing a two-dimensional transform space solution based on the work of Kharoufeh [8].

This thesis reduced the solution to one dimension by two separate methods. The first method was a direct method using an integrating factor to solve an ordinary differential equation. Previous work showed that the distribution satisfies the PDE system shown in Equation (3.6). By using the Laplace transform, the PDE system was converted into an ODE system with constant coefficients. Through the use of an integrating factor and the initial conditions, a solution was found for the cumulative distribution function in one-dimensional transform space.

The second method involved using the two-dimensional transform solution found in the existing literature. It was shown that the two-dimensional solution can be analytically inverted with respect to the first transform variable, leaving a solution in only one-dimensional transform space. Numerical inversion in a single dimension was then performed to produce approximations for the desired cumulative probability values.

Once these results were developed, this thesis gave a brief overview of some of the techniques for one- and two-dimensional numerical inversion of Laplace transforms. These techniques provide the basis for the MATLAB® algorithms which were used to produce numerical results in this work. Not only did this overview demonstrate the performance of numerical inversion, but it also helped to show the advantage of inverting in a single dimension rather than in two.

The next step in the thesis was to test the one-dimensional solution through numerical examples. An assortment of real-world scenarios were chosen to illustrate the utility of Markov reward processes. It is evident that these processes can be used to model a wide variety of systems, and that these systems may have any finite number of states. This work specifically examined MRP's in reliability theory, vehicular traffic flow theory, and queueing theory. Once each system was modelled as a MRP, Equation (3.20) was used to find exact solutions in one-dimensional transform space for the first passage time distributions. These were then numerically inverted using Abate and Whitt's [1] one-dimensional inversion algorithm to produce numerical approximations in the time domain. In several cases, these results were compared to results garnered from the two-dimensional technique as well as with Monte-Carlo simulation results. This comparison with the simulation clearly showed that both the one- and two-dimensional methods produce favorable results.

The major contribution of this work was the development of Theorem 3.1, which provided a solution for the distribution of first passage times in one-dimensional transform space. To utilize this theorem, it is necessary to provide three inputs: the

infinitesimal generator matrix for the governing CTMC, the diagonal matrix of reward rates, and the state of the CTMC at time $t = 0$. These are the same three inputs necessary to find solutions using the existing two-dimensional technique. Therefore, no additional knowledge of the system is required to reduce the dimensionality of the problem from two to one. In fact, less knowledge of the system is required since two-dimensional numerical inversion algorithms typically require the analyst to supply numerous parameters based on the estimated time-domain function. The standard one-dimensional numerical inversion algorithms require no such parameters.

Perhaps the greatest advantage, however, to finding solutions in one-dimensional transform space rather than two, is the significant improvement in computational time. As the size of the governing CTMC grows, the computation time in the two-dimensional technique increases rapidly. The one-dimensional technique shows no such increase. In the examples presented in Chapter 4, there was no noticeable difference between the time required to solve a 2-state problem and the time required to solve the 20-state problem. On the other hand, with the existing method, the computer required approximately 1 second to solve for a single cumulative probability value in the 2-state problem, and would need about 25 minutes to solve for a single cumulative probability value in the 20-state problem. Thus, Theorem 3.1 proved to be useful in the presented sample problems, and would be extremely convenient for real-world systems whose dimensionality might exceed 100 states.

Techniques for finding the moments of first passage times based on the two-dimensional solution can be found in the existing literature [8]. However, it is possible that a quicker and more accurate method may exist as a result of Theorem 3.1. Additionally, the results in this work, as well as any future analysis of moments, could be applied to larger problems (in excess of 100 states), utilizing real-world data. This would clearly illustrate how accurately MRP's can model real systems, and also show the enormous time savings associated with one-dimensional analysis.

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14. ABSTRACT

In this thesis, the problem of computing the cumulative distribution function (cdf) of the random time required for a system to first reach a specified reward threshold when the rate at which the reward accrues is controlled by a continuous-time stochastic process is considered. This random time is a type of first passage time for the cumulative reward process. The major contribution of this work is a simplified, analytical expression for the Laplace-Stieltjes transform of the cdf in one dimension rather than two. The result is obtained using two techniques: i) by converting an existing partial differential equation to an ordinary differential equation with a known solution, and ii) by inverting an existing two-dimensional result with respect to one of the dimensions. The results are applied to a variety of real-world operational problems using one-dimensional numerical Laplace inversion techniques and compared to solutions obtained from numerical inversion of a two-dimensional transform, as well as those from Monte-Carlo simulation. Inverting one-dimensional transforms is computationally more expedient than inverting two-dimensional transforms, particularly as the number of states in the governing Markov process increases. The numerical results demonstrate the accuracy with which the one-dimensional result approximates the first passage time probabilities in a comparatively negligible amount of time.

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